Tight Bounds for Tseitin Formulas

Dmitry Itsykson [1] Artur Riazanov [1, 2] Petr Smirnov [1, 3]

[1]St. Petersburg Department of V.A. Steklov Mathematical Institute of RAS, Russia

[2]The Henry and Marylin Taub Faculty of Computer Science, Technion, Israel

[3]HSE University at Saint Petersburg, Russia

The 25th International Conference
on Theory and Applications of Satisfiability Testing
August 3, 2022

Table of Contents

Proof Complexity

Proof systems are ways to certify formula unsatisfiability (e.g. natural deduction)

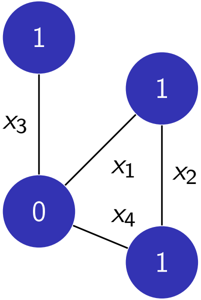
1. Polynomially bounded proof system does not exist $\implies \mathrm{NP} \neq \mathrm{coNP}$
   Cook's program [Cook, Reckhow; 1979]
2. SAT-solvers are equivalent to proof systems
   DPLL-solvers — tree-like resolution
   CDCL-solvers — general resolution

Automatability

- A proof system Π is **automatable** if there is an algorithm that finds a refutation of any formula ϕ in time $T \leq \text{poly}(|\phi|, S) = \exp(\log|\phi| + \log S)$, where S is the size of the shortest Π-refutation of ϕ.
- Many classical proof systems (resolution, cutting planes, and others) are not automatable unless $\mathrm{P} = \mathrm{NP}$.
- There are weakened notions of automatability:
  quasi-automatability: $T \leq \exp(\text{poly}(\log|\phi|, \log S))$
  almost automatability: $T \leq \exp(\log|\phi| \cdot \log S)$ (introduced in this work)
- Non-automatability results hold for the class of *all* CNF formulas. We may consider automatability on some important formula classes.

Tseitin Formulas

- Tseitin formula $\mathrm{T}(G, c)$ is defined for a graph
  - every edge is labeled with a variable
  - every vertex has a 0-1 label: $c : V \to \{0, 1\}$
  - $\mathrm{T}(G, c)(\vec{x}) = 1 \iff$
    $$\bigwedge_{v \in V} \left(\sum_{e \text{ is incident to } v} x_e = c(v) \bmod 2 \right) = \bigwedge_{v \in V} P_v$$
- A Tseitin formula is satisfiable iff for every connected component the sum of labels is even
- Unsatisfiable Tseitin formulas are classical hard instances for proof systems



$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_3 + x_4 = 0 \\ x_2 + x_4 = 1 \\ x_3 = 1 \end{cases}$$

Automatability on Tseitin formulas

- To prove that Π is automatable on Tseitin formulas, we may prove a constructive upper bound and a matching lower bound *for all Tseitin formulas*

Automatability on Tseitin formulas

- To prove that Π is automatable on Tseitin formulas, we may prove a constructive upper bound and a matching lower bound *for all Tseitin formulas*
- Classical lower bounds works only for specific graphs (expanders, grids)

Automatability on Tseitin formulas

- To prove that Π is automatable on Tseitin formulas, we may prove a constructive upper bound and a matching lower bound *for all Tseitin formulas*
- Classical lower bounds works only for specific graphs (expanders, grids)
- Grid Minor Theorem: any graph contains a grid minor $t \times t$, where $t = \Omega\left(\mathrm{tw}(G)^{\lambda}\right)$. Known for $\lambda = 0.1$, necessary $\lambda \leq 0.5$. [Robertson, Seymour, 1986; Chuzhoy, 2015]
- Gives lower bounds for all Tseitin formulas in several proof systems:
  Resolution: $S \geq \exp(\mathrm{tw}(G)^{\lambda})$
  OBDD(∧, reordering): $S \geq \exp(\mathrm{tw}(G)^{\lambda})$ [Glinskih, Itsykson, 2019]
  Depth-d Frege: $S \geq \exp(\mathrm{tw}(G)^{\Omega(1/d)})$ for $d \leq \frac{C \log n}{\log \log n}$ [GIRS, 2019]
- Very general approach, gives bounds far from optimal.

Automatability on Tseitin formulas

- To prove that Π is automatable on Tseitin formulas, we may prove a constructive upper bound and a matching lower bound *for all Tseitin formulas*
- Classical lower bounds works only for specific graphs (expanders, grids)
- Grid Minor Theorem: any graph contains a grid minor $t \times t$, where $t = \Omega\left(\mathrm{tw}(G)^{\lambda}\right)$. Known for $\lambda = 0.1$, necessary $\lambda \leq 0.5$. [Robertson, Seymour, 1986; Chuzhoy, 2015]
- Gives lower bounds for all Tseitin formulas in several proof systems:
  Resolution: $S \geq \exp(\mathrm{tw}(G)^{\lambda})$
  OBDD($\wedge$, reordering): $S \geq \exp(\mathrm{tw}(G)^{\lambda})$ [Glinskih, Itsykson, 2019]
  Depth-d Frege: $S \geq \exp(\mathrm{tw}(G)^{\Omega(1/d)})$ for $d \leq \frac{C \log n}{\log \log n}$ [GIRS, 2019]
- Very general approach, gives bounds far from optimal.
- We prove better lower bounds for Regular Resolution and OBDD($\wedge$, reordering). In both cases, we do it by (different) reductions from *satisfiable* Tseitin formulas.

Table of Contents

Regular Resolution

- Resolution refutation of a CNF formula ϕ uses **resolution rule:**

$$\frac{C \vee x, D \vee \neg x}{C \vee D} \quad \text{(eliminates } x\text{)}$$

- A refutation of ϕ is a sequence of clauses $C_1, C_2, \ldots, C_s$ such that
  - for every i, C_i is either
    - a clause of ϕ or
    - obtained by the resolution rule from previous clauses
  - C_s is an empty clause (i.e. identically false)
- **Regular** resolution: for any path in the proof graph, all eliminated variables are different.

Automatability of Regular Resolution on Tseitin formulas

- An upper bound is given by BWBATP algorithm [Alekhnovich, Razborov, 2011]: $T \leq \exp(\mathrm{tw}(G) \cdot \Delta + \log |V|)$.
- Automatability: $T \leq \exp(\log |\phi| + \log S)$,
  Almost automatability: $T \leq \exp(\log |\phi| \cdot \log S)$,
  Quasi-automatability: $T \leq \exp(\mathrm{poly}(\log |\phi|, \log S))$

	$\log S \geq \Omega(\ldots)$	$\Delta = \mathcal{O}(1)$	Δ is arbitrary
Grid Minor Theorem	$\mathrm{tw}(G)^\lambda$	quasi-aut.	quasi-aut.
IRSS, 2019	$\mathrm{tw}(G)/\log \lvert V \rvert$	almost aut.	quasi-aut.
De Colnet, Mengel, SAT 2021	$\mathrm{tw}(G)/\Delta$	automatable	quasi-aut.
New result	$\mathrm{tw}(G)$	automatable	almost aut.

Previous Results

Theorem (Itsykson, Riazanov, Sagunov, S., 2019)

Let $G = (V, E)$ be a graph, $\mathrm{T}(G, c)$ be unsatisfiable.
Then $\mathrm{RegRes}(\mathrm{T}(G, c)) \geq \exp(\mathrm{tw}(G)/\log|V|)$.

1. RegRes for $\mathrm{T}(G, c) \longrightarrow$ 1-BP for $\mathrm{T}(G, c')$ of size $S^{\mathcal{O}(\log|V|)}$, where $\mathrm{T}(G, c')$ is satisfiable
2. 1-$\mathrm{BP}(\mathrm{T}(G, c')) \geq \exp(\mathrm{tw}(G))$

Previous Results

Theorem (Itsykson, Riazanov, Sagunov, S., 2019)

Let $G = (V, E)$ be a graph, $\mathrm{T}(G, c)$ be unsatisfiable.
Then $\mathrm{RegRes}(\mathrm{T}(G, c)) \geq \exp(\mathrm{tw}(G)/\log|V|)$.

1. RegRes for $\mathrm{T}(G, c) \longrightarrow$ 1-BP for $\mathrm{T}(G, c')$ of size $S^{\mathcal{O}(\log|V|)}$, where $\mathrm{T}(G, c')$ is satisfiable
2. $1\text{-}\mathrm{BP}(\mathrm{T}(G, c')) \geq \exp(\mathrm{tw}(G))$

Theorem (de Colnet, Mengel, 2021)

Let $G = (V, E)$ be a graph with maximal degree Δ and $\mathrm{T}(G, c)$ be unsatisfiable.
Then $\mathrm{RegRes}(\mathrm{T}(G, c)) \geq \exp(\mathrm{tw}(G)/\Delta)/|V|$.

1. RegRes for $\mathrm{T}(G, c) \longrightarrow \mathrm{DNNF}$ for $\mathrm{T}(G, c')$ of size $S|V|$, where $\mathrm{T}(G, c')$ is satisfiable
2. $\mathrm{DNNF}(\mathrm{T}(G, c')) \geq \exp(\mathrm{tw}(G)/\Delta)$

New Lower Bound for Regular Resolution

Theorem

Let $G = (V, E)$ *be a graph and* $\mathrm{T}(G, c)$ *be unsatisfiable.*
Then $\mathrm{RegRes}(\mathrm{T}(G, c)) \geq \exp(\mathrm{tw}(G))$.

1. RegRes for $\mathrm{T}(G, c) \longrightarrow \mathrm{DNNF}$ for $\mathrm{T}(G, c')$ of size $S|V|$, where $\mathrm{T}(G, c')$ is satisfiable
2. $\mathrm{DNNF}(\mathrm{T}(G, c')) \geq \exp(\mathrm{tw}(G))$

Moreover, we prove that this is an exact size characterization of DNNF computing Tseitin formulas:

Theorem

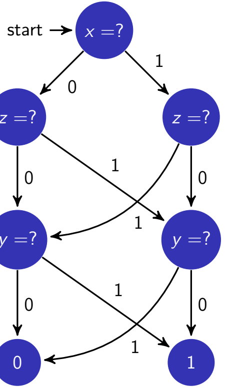
$\mathrm{DNNF}(\mathrm{T}(G, c')) = \exp(\Theta(\mathrm{tw}(G)))$.

So such a reduction to DNNF can not give better lower bounds.

Table of Contents

OBDD

- OBDD
  - π-OBDD represents a Boolean function f
  - it is a DAG with nodes labeled with variables
  - each node has two outgoing edges, one is labeled with 0 and the other is labeled with 1
  - the value of $f(\vec{x})$ determined by the label of the sink at the end of the path corresponding to $\vec{x}$
  - for every path, variables are appeared in the same order π
- If D_1 and D_2 are OBDDs in the same order, we can construct $D_1 \wedge D_2$ in time $\mathcal{O}(|D_1||D_2|)$
- If D_1 and D_2 are OBDDs in the same order, we can check them for equivalence in time $\mathcal{O}(|D_1||D_2|)$
- If D_1 is a π-OBDD representing f, we can construct σ-OBDD D_2 representing the same f in time $\mathcal{O}(|D_1||D_2|)$



$f(x,y,z) = x \oplus y \oplus z$, order π is (x,z,y)

OBDD($\wedge$, reordering)

- An OBDD($\wedge$, reordering) refutation of ϕ is a sequence of OBDDs $D_1, D_2, \ldots, D_s$ such that
  - for every i, D_i is either
    - represents a clause of ϕ, or
    - obtained by **conjunction** of two previous OBDDs D_j and D_k $(j, k < i)$ having the same variable order, or
    - obtained by **changing variable order** of a previous OBDD D_j $(j < i)$.
  - D_s represents identically false function.
- Note that every OBDD in a refutation corresponds to a subset of clauses of ϕ.
- The size of a refutation is the sum of the sizes of all OBDDs in it.

Automatability of OBDD(∧, reordering) on Tseitin formulas

- We prove a constructive upper bound: $T \leq \exp(\mathrm{tw}(G) \log |\mathrm{T}(G, c)|)$.
- And a new lower bound:

	$\log S \geq \Omega(\ldots)$	Δ is arbitrary
Glinskih, Itsykson, 2019	$\mathrm{tw}(G)^{\lambda}$	quasi-automatable
New result	$\mathrm{tw}(G)$	almost automatable

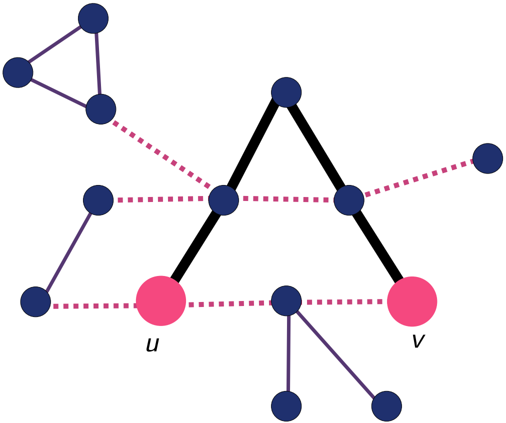
The Plan of the Proof

Consider the minimal refutation of unsatisfiable $\mathrm{T}(G, c)$.

- Consider its last step. It is the conjunction: $D_1 \wedge D_2 = D$, where D represents unsatisfiable $\mathrm{T}(G, c)$ and is identically false.
- Choose partial assignments α_1 and α_2, such that if $F_1 = D_1|_{\alpha_1}$ and $F_2 = D_2|_{\alpha_2}$, then $F_1 \wedge F_2 = D'$ represents *satisfiable* $\mathrm{T}(G', c')$.
- Note that $|D'| \leq |F_1| \cdot |F_2|$.
- If we can estimate $|D'| \geq S$, then $|F_1|$ or $|F_2|$ is at least $\Omega(\sqrt{S})$, and the same for D_1 and D_2.
- We have such an estimation: $|D'| \geq \exp(\mathrm{tw}(G'))$ [IRSS, 2019].

The Plan of the Proof (cont.)

- D_1 does not contain a clause for a vertex $v \in V$, D_2 does not contain a clause for a vertex $u \in V$.
- We want to remove v, u and some path connecting them such that the rest part of Tseitin formula is satisfiable.
- Choose substitutions carefully.
- Preserve treewidth: In a 2-connected graph, for any vertices v and u, there is a path p such that $\mathrm{tw}(G \setminus V(p)) \geq \Omega(\mathrm{tw}(G))$ [Robertson, Seymour, Thomas, 1994].
- $G' = G \setminus V(p)$, and D' computing $\mathrm{T}(G', c')$ has size at least $\exp(\Omega(\mathrm{tw}(G)))$.



The Plan of the Proof (cont.)

- D_1 does not contain a clause for a vertex $v \in V$, D_2 does not contain a clause for a vertex $u \in V$.
- We want to remove v, u and some path connecting them such that the rest part of Tseitin formula is satisfiable.
- Choose substitutions carefully.
- Preserve treewidth: In a 2-connected graph, for any vertices v and u, there is a path p such that $\mathrm{tw}(G \setminus V(p)) \geq \Omega(\mathrm{tw}(G))$ [Robertson, Seymour, Thomas, 1994].
- $G' = G \setminus V(p)$, and D' computing $\mathrm{T}(G', c')$ has size at least $\exp(\Omega(\mathrm{tw}(G)))$.

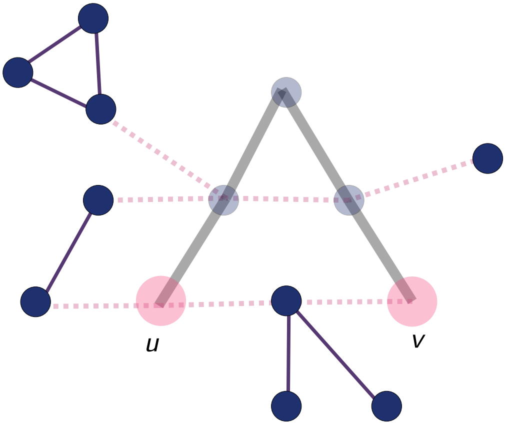


Table of Contents

Open Questions

- Automatability for RegRes: $\mathrm{RegRes}(\mathrm{T}(G, c)) \geq \exp(\mathrm{tw}(L(G)))$?
- Automatability for OBDD($\wedge$, reordering)?
- Prove a lower bound for all Tseitin formulas for unrestricted resolution. Now we have only a bound using Grid Minor Theorem.
- Urquhart's conjecture: Regular resolution polynomially simulates general resolution on Tseitin formulas.